Linear Algebra 21341
Homework 1

Due Sunday Jan 25, 11:59pm

1. Let $\mathbb{R}_+^*$ be the set of positive real numbers. Define the operations
$$u \oplus v := uv \quad \text{and} \quad \lambda \odot u := u^\lambda,$$
for $u, v \in \mathbb{R}_+^*$ and $\lambda \in \mathbb{R}$. Verify that $(\mathbb{R}_+^*, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

2. Prove that a vector space E over a field $\mathbb{K}$ is never the union of two proper subspaces.

3. Let $\mathbb{R}^\mathbb{R}$ denote the vector space of all real-valued functions on $\mathbb{R}$. Among the following subsets, determine which ones are subspaces of $\mathbb{R}^\mathbb{R}$:
   (1) $E_1 = \{f \in \mathbb{R}^\mathbb{R} \mid f(1) = 0\}$.
   (2) $E_2 = \{f \in \mathbb{R}^\mathbb{R} \mid f(0) = 1\}$.
   (3) E_3, the set of increasing functions from $\mathbb{R}$ to $\mathbb{R}$.
   (4) E_4, the set of decreasing functions from $\mathbb{R}$ to $\mathbb{R}$.
   (5) E_5, the set of monotone functions from $\mathbb{R}$ to $\mathbb{R}$.
   (6) E_6, the set of even functions from $\mathbb{R}$ to $\mathbb{R}$.
   (7) E_7, the set of odd functions from $\mathbb{R}$ to $\mathbb{R}$.
   (8) E_8, the set of 2π-periodic functions.
   (9) E_9, the set of periodic functions.

4. Let E be the set of convergent real sequences, F the set of real sequences converging to 0, and G the set of constant real sequences.
   (a) Prove that E, F, and G are subspaces of $\mathbb{R}^\mathbb{N}$.
   (b) Prove that $E = F \oplus G$.

5. Let F, G, and H be subspaces of a vector space E over a field $\mathbb{K}$.
   (a) What do you think of the following statement?
$$F + G = F \quad \text{if and only if} \quad F \supset G.$$
   (b) What do you think of the following statement?
$$F + G = F + H \implies G = H?$$

6. Let F, G, and H be subspaces of a vector space E over a field $\mathbb{K}$ such that
$$F + H = G + H, \quad F \cap H = G \cap H, \quad \text{and} \quad F \subset G.$$
Prove that $F = G$.