Linear Algebra 21341
Homework 2

Due Sunday Feb 1, 11:59pm

1. Let F and G be subspaces of a vector space E. Prove that

$$F \cap G = F + G \iff F = G.$$

2. Let E denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ for which there exists a real sequence $s = (s_n)_{n \in \mathbb{N}_{\geq 1}}$, called an *adapted sequence* to f, such that for all $n \in \mathbb{N}_{\geq 1}$ and all $x \in \mathbb{R}$,

$$\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) = s_n\, f(nx).$$

   (a) Show that constant functions belong to E.

   (b) Let $A : \mathbb{R} \to \mathbb{R}$ be the function defined by $A(x) = x - \frac{1}{2}$. Show that A is an element of E.

   (c) Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Is E a vector subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$?

3. Compare $\operatorname{span}(A \cap B)$ and $\operatorname{span}(A) \cap \operatorname{span}(B)$. In particular, determine whether one is contained in the other, and whether equality holds in general.

4. Consider the vectors $u = (1, 1, 1)$ and $v = (1, 0, -1)$ in $\mathbb{R}^3$.

   Show that

$$\operatorname{span}(u, v) = \{(2\alpha,\ \alpha + \beta,\ 2\beta) \mid \alpha, \beta \in \mathbb{R}\}.$$

5. Let E be a vector space and let F, G, and H be three subspaces of E. Show that F, G, and H are in direct sum if and only if

$$F \cap G = \{0\} \quad \text{and} \quad (F + G) \cap H = \{0\}.$$