Linear Algebra 21341
Homework 3

Due Monday Feb 9, 11:59pm

1. **Sequences and vector spaces.**

   Let
   $$E = \left\{ u = (u_n)_{n\in\mathbb{N}} \;\middle|\; \forall n \in \mathbb{N},\ 8u_{n+3} - 12u_{n+2} + 6u_{n+1} - u_n = 0 \right\}.$$

   Let the sequences $i = (i_n)_{n\in\mathbb{N}}$, $j = (j_n)_{n\in\mathbb{N}}$, and $k = (k_n)_{n\in\mathbb{N}}$ be defined by
   $$i_n = \left(\tfrac{1}{2}\right)^n, \qquad j_n = n\left(\tfrac{1}{2}\right)^n, \qquad k_n = n^2\left(\tfrac{1}{2}\right)^n, \qquad (n \in \mathbb{N}).$$

   (a) Show that E is a vector subspace of $\mathbb{R}^{\mathbb{N}}$.

   (b) (i) Show that $i \in E$, $j \in E$, and $k \in E$.

   (ii) Show that (i, j, k) is a linearly independent family in E.

   (c) Let $u = (u_n)_{n\in\mathbb{N}} \in E$. Define the sequence $w = (w_k)_{k\in\mathbb{N}}$ by
   $$w_k = 2^{k+1}u_{k+1} - 2^k u_k.$$

   Assume that there exist $\lambda, \mu \in \mathbb{R}$ such that
   $$w_k = \lambda + \mu k \qquad (k \in \mathbb{N}).$$

   (i) For $n \geq 1$, compute in two different ways the sum
   $$S_{n-1} = \sum_{k=0}^{n-1} w_k.$$

   (ii) Deduce that u is a linear combination of the sequences i, j, and k.

   (d) Using the previous results, determine a basis of E.

   (e) We now justify the result assumed in part (c).
   Let $u = (u_n)_{n\in\mathbb{N}} \in E$ and define $w = (w_k)_{k\in\mathbb{N}}$ by
   $$w_k = 2^{k+1}u_{k+1} - 2^k u_k.$$

   (i) Show that for all $k \in \mathbb{N}$,
   $$w_{k+2} - 2w_{k+1} + w_k = 0.$$

   (ii) Deduce the expression of w_k as a function of k.

(f) Define

$$F = \left\{ u = (u_n)_{n\in\mathbb{N}} \in E \;\middle|\; u_0 - 2u_1 = 0 \right\}.$$

(i) Show that F is a vector subspace of E.

(ii) Show that $i \in F$.

(iii) Let $u = (u_n)_{n\in\mathbb{N}} \in F$. Show that there exists a unique $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that

$$u_n = \alpha i_n + \beta j_n + \gamma k_n \qquad (n \in \mathbb{N}),$$

and prove that $\gamma = -\beta$.

(iv) Deduce a basis of F.

2. **Linear independence of families of functions.**

Show that in the $\mathbb{R}$-vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, the following families are linearly independent:

(a) $(f_\lambda)_{\lambda\in\mathbb{R}}$, where

$$f_\lambda : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto e^{\lambda x}.$$

(b) $(f_\lambda)_{\lambda\in\mathbb{R}_+}$, where

$$f_\lambda : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \cos(\lambda x).$$

(c) $(f_\lambda)_{\lambda\in\mathbb{R}}$, where

$$f_\lambda : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto |x - \lambda|.$$