Linear Algebra 21341
Homework 4

Due Wednesday Feb 18, 11:59pm

1. **Convergent sequences.**

   Let E be the set of all convergent real sequences, F the set of real sequences whose limit is 0, and G the set of constant sequences.

   (1) Show that E, F, and G are vector subspaces of $\mathbb{R}^{\mathbb{N}}$.

   (2) Show that
   $$E = F \oplus G.$$

2. **Functions on $\mathbb{R}$.**

   Let $E = \mathbb{R}^{\mathbb{R}}$. Define
   $$F = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(0) + f(1) = 0\}$$
   and let G be the set of constant functions on $\mathbb{R}$.

   (1) Show that F and G are vector subspaces of E.

   (2) Show that F and G are complementary in E (i.e., $E = F \oplus G$).

3. **Differential equations.**

   Consider the following differential equations:
   $$(E):\ y''' - y = 0, \qquad (F):\ y'' + y' + y = 0, \qquad (G):\ y' - y = 0.$$

   Let E, F, and G be the respective sets of real-valued solutions of (E), (F), and (G). The solutions are all of class C^∞ on $\mathbb{R}$. Let $C^\infty(\mathbb{R}, \mathbb{R})$ denote the set of infinitely differentiable real-valued functions on $\mathbb{R}$.

   (1) Show that E is a vector subspace of $C^\infty(\mathbb{R}, \mathbb{R})$.

   (2) Show that $F \subset E$ and $G \subset E$.

   (3) Find the solutions of (F) and (G).

   (4) Show that F and G are vector subspaces of E and give a generating family for each.

   (5) (a) Let $y \in E$. Define
   $$y_1 = 2y - y' - y'', \qquad y_2 = y + y' + y''.$$
   Show that $y_1 \in F$ and $y_2 \in G$.

(b) Show that F and G are complementary in E.

(6) Deduce the set of solutions of (E).

4. **Kernel and image.**

Let E be a vector space over $\mathbb{K}$ and let a be a nonzero element of $\mathbb{K}$. Let $f \in \mathcal{L}(E)$ satisfy

$$f^3 - 3af^2 + a^2 f = 0.$$

Is it true that $\ker(f)$ and $\mathrm{Im}(f)$ are complementary in E?