Linear Algebra 21-341
Homework 5

Due Monday, March 2, 11:59pm

1. **A nilpotent endomorphism in dimension 3.**
   Let E be a 3-dimensional vector space over $\mathbb{K}$ and let $f \in \mathcal{L}(E)$ satisfy
   $$f^2 \neq 0 \qquad \text{and} \qquad f^3 = 0.$$
   Show that there exists a basis of E in which the matrix of f is
   $$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

2. **A nilpotent endomorphism of maximal index.**
   Let E be an n-dimensional vector space over $\mathbb{K}$ with $n \geq 1$, and let $f \in \mathcal{L}(E)$ satisfy
   $$f^n = 0 \qquad \text{and} \qquad f^{n-1} \neq 0.$$
   Show that there exists a basis $\mathcal{B}$ of E such that
   $$\mathrm{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

3. **A matrix identity forcing commutativity.**
   Let $A, B \in M_n(\mathbb{K})$ satisfy
   $$AB = A + B.$$
   Prove that A and B commute, i.e. $AB = BA$.

4. **When does equality hold for the rank of a sum?**
   Let E and F be finite-dimensional vector spaces over $\mathbb{K}$ and let $f, g \in \mathcal{L}(E, F)$.
   Prove that
   $$\mathrm{rank}(f+g) = \mathrm{rank}(f) + \mathrm{rank}(g)$$
   if and only if
   $$\begin{cases} \mathrm{Im}(f) \cap \mathrm{Im}(g) = \{0\}, \\ \ker(f) + \ker(g) = E. \end{cases}$$

5. **Iterated kernels and images.**

Let E be a finite-dimensional real vector space with $\dim(E) = n \geq 1$, and let $f \in \mathcal{L}(E)$.

(A) Sequence of iterated kernels.

1. Show that for every $i \in \mathbb{N}$,
$$\ker(f^i) \subset \ker(f^{i+1}).$$
2. Assume that there exists $r \in \mathbb{N}$ such that
$$\ker(f^r) = \ker(f^{r+1}).$$
Show that for every $i \geq r$,
$$\ker(f^i) = \ker(f^{i+1}).$$
3. For $i \in \mathbb{N}$, set $d_i = \dim(\ker(f^i))$.
   (a) Show that the sequence $(d_i)_{i\in\mathbb{N}}$ is monotone.
   (b) Show that there exists $r \in \{0, 1, \ldots, n\}$ such that $d_r = d_{r+1}$.
   (c) Deduce that the sequence $(d_i)_{i\in\mathbb{N}}$ is stationary.

(B) Sequence of iterated images.

1. Show that for every $j \in \mathbb{N}$,
$$\mathrm{Im}(f^{j+1}) \subset \mathrm{Im}(f^j).$$
2. Assume that there exists $s \in \mathbb{N}$ such that
$$\mathrm{Im}(f^{s+1}) = \mathrm{Im}(f^s).$$
Show that for every $j \geq s$,
$$\mathrm{Im}(f^{j+1}) = \mathrm{Im}(f^j).$$
3. For $j \in \mathbb{N}$, set $m_j = \dim(\mathrm{Im}(f^j))$.
   (a) Show that the sequence $(m_j)_{j\in\mathbb{N}}$ is monotone.
   (b) Let r be as in part **(A)**.3(b). Show that $m_{r+1} = m_r$.
   (c) Deduce that the sequence $(m_j)_{j\in\mathbb{N}}$ is stationary.

Appendix: Solution to a previous problem (Kernel and Image)

Let E be a vector space over $\mathbb{K}$ and let $a \in \mathbb{K}$ with $a \neq 0$. Let $f \in \mathcal{L}(E)$ satisfy
$$f^3 - 3af^2 + a^2 f = 0.$$

We show that $\ker(f)$ and $\mathrm{Im}(f)$ are complementary in E.

Step 1: $\ker(f) \cap \mathrm{Im}(f) = \{0\}$**.** Let $y \in \ker(f) \cap \mathrm{Im}(f)$. Then $y = f(x)$ for some $x \in E$ and $f(y) = 0$. Apply the polynomial identity to x:
$$f^3(x) - 3af^2(x) + a^2 f(x) = 0.$$

Since $f(x) = y$ and $f^2(x) = f(y) = 0$, this gives $a^2 y = 0$. Because $a \neq 0$ (hence $a^2 \neq 0$), we obtain $y = 0$.

Step 2: $E = \ker(f) + \mathrm{Im}(f)$**.** For any $x \in E$, define

$$x_K := \frac{1}{a^2}\big(a^2 x + f^2(x) - 3af(x)\big), \qquad x_I := \frac{1}{a^2}\big(-f^2(x) + 3af(x)\big).$$

Then clearly $x = x_K + x_I$. Moreover,

$$f(x_K) = \frac{1}{a^2}\big(a^2 f(x) + f^3(x) - 3af^2(x)\big) = 0$$

by the assumed identity, so $x_K \in \ker(f)$; and

$$x_I = f\left(\frac{1}{a^2}(-f(x) + 3ax)\right) \in \mathrm{Im}(f),$$

so $x_I \in \mathrm{Im}(f)$. Hence $E = \ker(f) + \mathrm{Im}(f)$.

Conclusion. Since $\ker(f) \cap \mathrm{Im}(f) = \{0\}$ and $E = \ker(f) + \mathrm{Im}(f)$, we conclude

$$E = \ker(f) \oplus \mathrm{Im}(f).$$

Let E be a vector space over $\mathbb{K}$ and let a be a nonzero element of $\mathbb{K}$. Let $f \in \mathcal{L}(E)$ satisfy

$$f^3 - 3af^2 + a^2 f = 0.$$

Is it true that $\ker(f)$ and $\mathrm{Im}(f)$ are complementary in E?