Linear Algebra 21341
Homework 6

Due Monday March 23, 11:59pm

1. **Examples of matrices similar to their inverse.**

   Throughout this problem, let E be a 3-dimensional real vector space. For an endomorphism u of E and an integer $n \geq 1$, we denote by u^n the n-fold composition

   $$u^n = \underbrace{u \circ u \circ \cdots \circ u}_{n \text{ times}}.$$

   We denote by $M_3(\mathbb{R})$ the vector space of 3×3 real matrices, by $GL_3(\mathbb{R})$ the group of invertible matrices in $M_3(\mathbb{R})$, and by I_3 the identity matrix. We denote by 0 the zero endomorphism, the zero matrix, and the zero vector, depending on the context.

   For two matrices $A, B \in M_3(\mathbb{R})$, we say that A is *similar* to B if there exists a matrix $P \in GL_3(\mathbb{R})$ such that

   $$A = P^{-1}BP.$$

   We recall that if $\mathcal{B}$ and $\mathcal{B}'$ are two bases of E, if P is the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{B}'$, and if u is an endomorphism of E with matrix A in the basis $\mathcal{B}'$ and matrix B in the basis $\mathcal{B}$, then

   $$A = P^{-1}BP.$$

   Part A. We write $A \sim B$ to mean that A is similar to B.

   1. Prove that the relation $\sim$ is an equivalence relation on $M_3(\mathbb{R})$.
   2. Prove that two matrices in $M_3(\mathbb{R})$ with different determinants cannot be similar.
   3. Let u be an endomorphism of E, and let i, j be natural numbers. Consider the map

      $$w : \ker(u^{i+j}) \to E, \qquad w(x) = u^j(x).$$

      (a) Show that $\mathrm{Im}(w) \subset \ker(u^i)$.

      (b) Deduce that

      $$\dim(\ker(u^{i+j})) \leq \dim(\ker(u^i)) + \dim(\ker(u^j)).$$

   4. Let u be an endomorphism of E such that

      $$u^3 = 0 \qquad \text{and} \qquad \mathrm{rank}(u) = 2.$$

      (a) Show that $\dim(\ker(u^2)) = 2$. (You may use Question 3(b) twice.)

(b) Show that one can find a nonzero vector $a \in E$ such that $u^2(a) \neq 0$, and deduce that the family

$$(u^2(a),\, u(a),\, a)$$

is a basis of E.

(c) Write the matrix U of u and the matrix V of $u^2 - u$ in this basis.

5. Let u be an endomorphism of E such that

$$u^2 = 0 \qquad \text{and} \qquad \operatorname{rank}(u) = 1.$$

(a) Show that one can find a nonzero vector $b \in E$ such that $u(b) \neq 0$.

(b) Justify the existence of a vector $c \in \ker(u)$ such that the family $(u(b), c)$ is linearly independent, and then show that the family

$$(b,\, u(b),\, c)$$

is a basis of E.

(c) Write the matrix U' of u and the matrix V' of $u^2 - u$ in this basis.

Part B. Let now $A \in M_3(\mathbb{R})$ be similar to a matrix of the form

$$T = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R}).$$

We want to prove that A is similar to its inverse A^{-1}.
Set

$$N = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix},$$

and let $P \in GL_3(\mathbb{R})$ be such that

$$P^{-1}AP = T = I_3 + N.$$

1. Explain why the matrix A is invertible.
2. Compute N^3 and show that

$$P^{-1}A^{-1}P = I_3 - N + N^2.$$

3. Assume in this question that $N = 0$. Show that A and A^{-1} are similar.
4. Assume in this question that $\operatorname{rank}(N) = 2$. Set

$$M = N^2 - N.$$

(a) Show that N is similar to the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and deduce, using Question A.4, a matrix similar to M.

(b) Compute M^3 and determine $\operatorname{rank}(M)$.
(c) Show that the matrices M and N are similar.
(d) Deduce that the matrices A and A^{-1} are similar.

5. Assume in this question that $\operatorname{rank}(N) = 1$. Set
$$M = N^2 - N.$$
Show that the matrices A and A^{-1} are similar.

6. **Example.** Let
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$
Let (a, b, c) be a basis of E, and let u be the endomorphism of E whose matrix in this basis is A.
   (a) Show that $\ker(u - \mathrm{id}_E)$ is a 2-dimensional subspace of E, and give a basis (e_1, e_2) of it.
   (b) Justify that the family (e_1, e_2, c) is a basis of E, and write the matrix of u in this basis.
   (c) Show that the matrices A and A^{-1} are similar.

7. Conversely, is every matrix in $M_3(\mathbb{R})$ that is similar to its inverse necessarily similar to a matrix of the form
$$T = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} ?$$

2. **A binomial-coefficient matrix.**

Let $n \in \mathbb{N}^*$ and consider the matrix
$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{n}{1} \\ 0 & 0 & \binom{2}{2} & \cdots & \binom{n}{2} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \binom{n}{n} \end{pmatrix} \in M_{n+1}(\mathbb{R}).$$

1. Prove that M is invertible.
2. Compute the inverse matrix M^{-1} explicitly.

Hint. Consider the linear operator
$$f : \mathbb{R}_n[X] \to \mathbb{R}_n[X], \qquad P(X) \mapsto P(X+1),$$
and interpret M as the matrix of f in the canonical basis
$$\mathcal{B} = (1, X, \ldots, X^n)$$
of $\mathbb{R}_n[X]$.

3. **A determinant recurrence.**

   Let $a, b \in \mathbb{K}$ with $a \neq b$. Define

$$A_n = \begin{pmatrix} a+b & ab & 0 & \cdots & 0 \\ 1 & a+b & ab & \ddots & \vdots \\ 0 & 1 & a+b & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & ab \\ 0 & \cdots & 0 & 1 & a+b \end{pmatrix} \in M_n(\mathbb{K}).$$

   Compute $\det(A_n)$.

4. **A determinant identity involving the trace.**

   Let $A \in M_n(\mathbb{K})$. Identify a matrix $M \in M_n(\mathbb{K})$ with its columns

$$(C_1, \ldots, C_n) \in \big(M_{n,1}(\mathbb{K})\big)^n.$$

   Define the map

$$f_A : M_n(\mathbb{K}) \to \mathbb{K}$$

   by

$$f_A(M) = \det(AC_1, C_2, \ldots, C_n) + \det(C_1, AC_2, \ldots, C_n) + \cdots + \det(C_1, C_2, \ldots, AC_n).$$

   Show that for every $M \in M_n(\mathbb{K})$,

$$f_A(M) = \operatorname{tr}(A)\det(C_1, \ldots, C_n) = \operatorname{tr}(A)\det(M).$$

5. **Trace and matrices.**

   (a) Let $A \in M_n(\mathbb{R})$ satisfy

$$\operatorname{tr}(AA^\top) = 0.$$

   Show that A is the zero matrix.

   (b) Let $A, B \in M_n(\mathbb{R})$ satisfy

$$\forall X \in M_n(\mathbb{R}), \qquad \operatorname{tr}(AX) = \operatorname{tr}(BX).$$

   What can you conclude about the matrices A and B?