Homework 8

Exercise. *Let $A \in M_5(\mathbb{R})$ be an invertible matrix such that*

$$\operatorname{Tr}(A) = 6 \qquad \text{and} \qquad A^3 - 3A^2 + 2A = 0.$$

1. *Show that there exists an annihilating polynomial of degree 2 for A.*
2. *Is the matrix A diagonalizable? What are its only possible eigenvalues?*
3. *Determine the algebraic multiplicity of each eigenvalue, and deduce the characteristic polynomial of A as well as its minimal polynomial.*

Exercise. *For $n \in \mathbb{N}^*$ and for $A, B \in M_n(\mathbb{R})$, define*

$$\langle A, B \rangle = \operatorname{Tr}(A^T B) + \operatorname{Tr}(A)\operatorname{Tr}(B).$$

1. *Show that the map $\langle \cdot, \cdot \rangle$ thus defined is an inner product on $M_n(\mathbb{R})$.*
2. *Let S_n (respectively A_n) denote the set of symmetric (respectively skew-symmetric) matrices in $M_n(\mathbb{R})$. If $S \in S_n$ and $A \in A_n$, show that*
$$\langle A, S \rangle = 0.$$
3. *Deduce the orthogonal complement of S_n with respect to this inner product.*
4. *For $M \in M_n(\mathbb{R})$, give the expression of $p_{S_n}(M)$, the orthogonal projection of M onto S_n.*

Exercise. *Let*

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

1. *What is its characteristic polynomial χ_A?*
2. *Compute the values $\chi_A(-1)$, $\chi_A(0)$, and $\chi_A(1)$, as well as the limits of $\chi_A(x)$ as $x \to \pm\infty$.*
3. *Is the matrix A diagonalizable over $\mathbb{R}$?*