Your Name

Andrew ID

Honor Statement

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________

- Unless stated otherwise, you must show and justify all of your work to receive credit.
- If you use scratch paper that you want graded, write your name on it and leave a note on the original page of the problem.
- The exam is graded out of 110 points, but your score will not be scaled proportionally.

Question	Points	Score
1	30	
2	20	
3	20	
4	20	
5	20	
Total	110	

Problem 1. (30 pts)
For $n \in \mathbb{N}^*$ and $A, B \in M_n(\mathbb{R})$, define

$$\langle A, B \rangle = \mathrm{Tr}(A^T B) + \mathrm{Tr}(A)\,\mathrm{Tr}(B).$$

1. (7.5 points) Show that $\langle \cdot, \cdot \rangle$ is an inner product on $M_n(\mathbb{R})$.

2. (7.5 points) Let S_n (respectively A_n) denote the set of symmetric (respectively skew-symmetric) matrices in $M_n(\mathbb{R})$. If $S \in S_n$ and $A \in A_n$, show that

$$\langle A, S \rangle = 0.$$

3. (7.5 points) Deduce the orthogonal complement of S_n with respect to this inner product.

4. (7.5 points) For $M \in M_n(\mathbb{R})$, determine the orthogonal projection of M onto S_n.

Problem 2. (20 pts)
The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined by

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and} \qquad F_n = F_{n-2} + F_{n-1} \quad \text{for } n \geq 2.$$

Prove that, for every nonnegative integer n,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Hint: You may define $T \in \mathscr{L}(\mathbb{R}^2)$ by

$$T(x,y) = (y, x+y).$$

Problem 3. (20 pts)
Let $n \geq 1$. For $A, B \in M_n(\mathbb{R})$, define
$$\langle A, B\rangle = \operatorname{tr}(A^T B).$$
Assume that $\langle \cdot, \cdot \rangle$ is an inner product on $M_n(\mathbb{R})$.
Show that for every pair of symmetric real matrices A and B (that is, $A^T = A$ and $B^T = B$), one has
$$\big(\operatorname{tr}(AB)\big)^2 \leq \operatorname{tr}(A^2)\operatorname{tr}(B^2).$$

Problem 4. (20 pts)
Consider the matrix

$$A = \begin{pmatrix} -9 & -5 & 16 \\ -2 & 0 & 4 \\ -6 & -3 & 11 \end{pmatrix} \in M_3(\mathbb{R}).$$

1. (10 points) Compute the characteristic polynomial of A and the eigenvalues of A.

2. (10 points) Deduce, without further computation, that A is diagonalizable over $\mathbb{R}$.

Problem 5. (20 pts)
Let $E = \mathbb{R}^3$, and let $u \in \mathscr{L}(E)$ satisfy

$$u^4 = u^2.$$

Assume moreover that -1 and 1 are eigenvalues of u. Show that u is diagonalizable over $\mathbb{R}$.