Your Name

Andrew ID

Honor Statement

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________

- Unless stated otherwise, you must show and justify all of your work to receive credit.
- If you use scratch paper that you want graded, write your name on it and leave a note on the original page of the problem.
- There are 110 points available on this exam. The final grade will be based on your overall score, but it will not necessarily be obtained by direct proportionality.

Question	Points	Score
1 (Quiz)	10	
2	20	
3	20	
4	20	
5	20	
6	20	
Total	110	

Problem 1. (10 pts) For each statement below, circle **True** or **False**. Do not justify your answer.

(a) The matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

are similar.

True / **False**

(b) The matrices

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

are similar.

True / **False**

(c) For all square matrices A, B, C of the same size,

$$\operatorname{tr}(ABC) = \operatorname{tr}(ACB).$$

True / **False**

(d) Similar matrices represent the same linear operator in different bases.

True / **False**

Problem 2. (20 pts)
Let E be an n-dimensional vector space over a field K, with $n \geq 1$, and let $f \in \mathscr{L}(E)$ satisfy

$$f^n = 0 \qquad \text{and} \qquad f^{n-1} \neq 0.$$

Show that there exists a basis $\mathscr{B}$ of E such that

$$\mathrm{Mat}_{\mathscr{B}}(f) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Problem 3. (20 pts)
Let $(e_1, \ldots, e_n)$ be a basis of a vector space E over a field $\mathbf{F}$, and let

$$a = \sum_{i=1}^{n} \alpha_i e_i.$$

Let $b_1, \ldots, b_n \in \mathbf{F}$. Compute

$$\det(e_1 + b_1 a,\ e_2 + b_2 a,\ \ldots,\ e_n + b_n a),$$

where the determinant is taken with respect to the basis $(e_1, \ldots, e_n)$.

Problem 4. (20 pts)
Consider the map $f : \mathbb{R}_3[X] \to \mathbb{R}_3[X]$ defined, for $a,b,c,d \in \mathbb{R}$, by

$$f(a+bX+cX^2+dX^3) = d + \frac{a+b+c}{2}X^2 + (d-b)X^3.$$

1. Prove that f is linear.

2. Compute the trace of f.

Problem 5. (20 pts)
Let E be a finite-dimensional real vector space and let $p \in \mathscr{L}(E)$ be a projection, i.e.

$$p \circ p = p.$$

Prove that

$$\operatorname{rank}(p) = \operatorname{tr}(p).$$

Problem 6. (20 pts)
Let F be a field and let $a,b,c,d,e,f,g \in F$. Show that

$$\begin{vmatrix} a & b & b \\ c & d & e \\ f & g & g \end{vmatrix} + \begin{vmatrix} a & b & b \\ e & c & d \\ f & g & g \end{vmatrix} + \begin{vmatrix} a & b & b \\ d & e & c \\ f & g & g \end{vmatrix} = 0.$$